# Comparison of Minimum-Propellant Guidance Laws for Impulsive and Bounded-Thrust Spacecraft

ALAIN L. KORNHAUSER* AND PAUL M. LION†
*Princeton University, Princeton, N. J.*

**The minimum-propellant deterministic guidance law for a bounded-thrust, constant jet exhaust velocity rocket is compared to the minimum-propellant impulsive-thrust guidance law. It is shown that the impulsive-thrust guidance law represents the limit of the bounded-thrust guidance law as the magnitude of the thrust acceleration becomes unbounded ($a \to \infty$) and the thrust duration goes to zero ($\tau \to 0$). This connection is of more than academic value in that it eases the interpretation of the bounded-thrust guidance law in terms of the much more easily understood impulsive-thrust case. Of importance is the insight provided towards the interpretation of the jump conditions and controllability condition in the one-burn case.**

## Nomenclature

$a$ = thrust acceleration
$\mathbf{A}$ = guidance gain matrix
$\mathbf{B},\mathbf{R}$ = sensitivity matrices
$c$ = jet exhaust velocity (assumed constant)
$\mathbf{C}$ = controllability matrix
$d[\ ]$ = noncontemporaneous variation
$\mathbf{D}$ = $(6 \times 3)$ allocation matrix
$\mathbf{f}$ = $\begin{bmatrix} v \\ g(r) \end{bmatrix}$
$\mathbf{F}$ = $\partial f/\partial x$
$\mathbf{g}$ = gravity vector
$\mathbf{G}$ = $(3 \times 3)$ gravity gradient matrix
$\mathbf{H}$ = Hamiltonian
$\mathbf{I}_n$ = $(n \times n)$ identity matrix
$J$ = cost functional
$k$ = impulse number
$m$ = mass
$p$ = magnitude of primer vector
$\mathbf{P}$ = $(3 \times 3)$ projector matrix (Rank = 2)
$\Theta$ = second-variation weighting matrix
$\mathbf{r}$ = $(3 \times 1)$ position vector
$t$ = time
$\mathbf{u}$ = control vector
$\mathbf{v}$ = $(3 \times 1)$ velocity vector
$\mathbf{x}$ = $(6 \times 1)$ state vector
$\partial$ = partial derivative
$\Gamma$ = $(6 \times 6)$ state-adjoint transition matrix
$\delta$ = contemporaneous first-order variation
$\eta$ = thrust vector direction
$\lambda$ = Lawden's primer vector
$\xi$ = dummy time variable
$\tau$ = thrust duration
$\Phi$ = $(6 \times 6)$ state-state transition matrix
$\psi$ = $(6 \times 1)$ adjoint vector
$\omega$ = $(3 \times 1)$ impulsive thrust vector
0 = null matrix
$\|$ = absolute value
$ = switch function
$\vartheta$ = order of

*Superscripts*

$T$ = transpose
$'$ = neighboring trajectory
$-$ = just before time $t$
$+$ = just after time $t$
$*$ = optimal value
= differentiation with time

*Subscripts*

$f$ = final time
$i$ = dummy index
$k$ = impulse or burn number
$n$ = number of burns or impulses
o = initial time
off = time of thrust-off
on = time of thrust-on

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* Assistant Professor of Civil and Geological Engineering. Member AIAA.
† Associate Professor of Aerospace and Mechanical Sciences. Member AIAA.

## Introduction

REFERENCES 1 and 2 develop the deterministic minimum-propellant guidance laws for both impulsive and bounded-thrust spacecrafts. The basic result of the analysis is that "correct-as-soon-as-possible" is in general a nonoptimal guidance rule-of-thumb on multiburn portions of a trajectory. Specifically, it is shown that the addition of corrective burns costs more than the augmentation of the preplanned nominal burns. As in the targeting problem, the impulsive-thrust guidance problem is much easier than the bounded-thrust problem both conceptually and numerically. In the impulsive-thrust case one has a closed form solution to the variational equations between impulses and simple jump conditions across impulses. In the bounded-thrust case the variational equations must be numerically integrated during burns. This greatly complicates the qualitative understanding of the effects of the possible guidance controls. However, a comparison of the bounded-thrust case to the impulsive-thrust case leads to much insight as to the actual corrections afforded in the bounded-thrust case. This is especially useful when dealing with coast-thrust trajectories. In such cases, the bounded-thrust trajectory is uncontrollable in the sense of applying the guidance laws developed for multiburn trajectories.

We will first present a review of the optimum guidance laws which were developed using the concept of neighboring externals. The bounded-thrust case will then be studied in the limit as the thrust bound goes to infinity. It will be shown that the variational equations, jump conditions, and degrees of freedom reduce to those of the impulsive-thrust case. This background will then enable us to analyze the uncontrollable one-burn case in the light of the well-known optimum guidance laws for the one-impulse case. The result is that the addition of new burns is less costly than the augmentation of the existing burn in performing some guidance corrections.

## Review of Minimum Propellant Guidance

In Appendix A á unified development of the minimum propellant guidance laws for impulsive and bounded-thrust rockets is presented. The development is based on References 1–3. It is presented for convenience to the reader and provides easy reference. The principle results of the development are linear feedback guidance laws for both impulsive- and bounded-thrust spacecrafts. These appear in the form of

$$\delta \mathbf{u} = \mathbf{A}(t_0)\,\delta \mathbf{x}(t_0) \qquad (1)$$

where $\delta\mathbf{u}$ is the guidance correction and $\delta\mathbf{x}(t_0)$ is the "initial" state variation. Because the matrix A can be calculated continuously along the trajectory, Eq. (1) is in feedback form. The matrix A can be calculated analytically without numerical integration or iteration in the impulsive-thrust case. In the bounded-thrust case numerical integration of the variational equations is required during the nominal-thrust periods.

The most important result of the development is the form of $\delta\mathbf{u}$. Out of all possible correction schemes it is shown that the only "admissible" corrections are: 1) small changes in timing, magnitude and direction of the nominal impulses in the impulsive-thrust case, and 2) small changes in the thrust-on/thrust-off times and small rotations of the nominal thrust vector in the bounded-thrust case. These corrections are found to increase the "cost" of the trajectory at most to second order while all other possibly useful corrections increase the cost to first order. Examples of trajectory corrections that are ruled-out are: the addition of small magnitude impulse or burns during nominal coasts and the throttling of the thrust magnitude. The trajectory correction afforded by the admissible impulsive-thrust guidance laws are easily understood conceptually whereas the bounded-thrust guidance is complicated by the continuous rotation of the thrust vector. We will attempt to circumvent this complication in the next section by analyzing the bounded-thrust case in the limit as the thrust becomes unbounded.

## Limiting Behavior of the Bounded Thrust Acceleration (BTA) Case

Neustadt[5] proved that the minimum-propellant impulsive-thrust trajectory represents the limit of minimum-propellant bounded-thrust trajectories as the thrust becomes unbounded. This basic result also implies that as the thrust becomes unbounded the duration of each burn goes to zero, $\tau_k \to 0$, its "equivalent change in velocity" $= a_0\tau_k \to |\omega_k|$ ($\omega_k$ is the impulsive-thrust vector normally denoted by $\Delta v_k$) and the thrust-on/thrust-off switch times converge on the impulse time, $t_{\text{on}_k} \to t_k$, $t_{\text{off}_k} \to t_k$. This behavior was used by Hazelrigg et al.,[6,7] Handelsman,[8] and others[9,10] to describe minimum-propellant bounded-thrust trajectories in terms of their corresponding minimum-propellant impulsive trajectory. In this section we will use the same approach to analyze the bounded-thrust minimum-propellant guidance law. For simplicity it is assumed that the propellant mass flow rate is sufficiently small to warrant the assumption that the spacecraft mass remains a constant. The qualitative results developed in this paper are unaffected by this assumption.

### Discontinuities in State and Variational State

For the case of impulsive thrust, the position vector $\mathbf{r}$ is always continuous in time, but the velocity vector $\mathbf{v}$ has a jump discontinuity at each impulse time $t_k$. This discontinuity propagates through higher order derivatives. It appears as an impulse in the acceleration, a doublet in the derivative of acceleration, etc.

In the bounded-thrust acceleration (BTA) case, both the position and velocity vectors are continuous. The acceleration has a jump discontinuity at each switch-time. Successively higher-order singularities appear in successively higher derivatives of the acceleration at each switch-time. Thus, as a result of the smoothing nature of the integration required over the thrust period, equivalent singularities in the state vector appear in one higher derivative in the BTA case than they appear in the impulsive-thrust case. These discontinuities are shown in Figs. 1 and 2 where

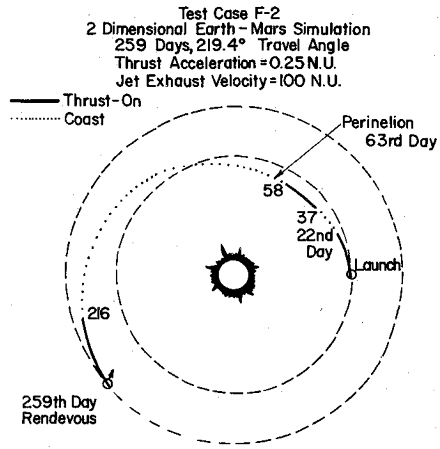


**Fig. 1 Heliocentric map of test case F-2 two-dimensional Earth-Mars rendezvous trajectory.**

$$\delta[\,\cdot\,] \equiv [\,\cdot\,](t) - [\,\cdot\,]'(t)$$

is the contemporaneous variation between two trajectories and

$$d[\,\cdot\,] \equiv [\,\cdot\,](t + dt) - [\,\cdot\,]'(t)$$

is the "skewed" (noncontemporaneous) variation.

Of primary interest to the guidance problem is the form of the variational state equations (A8). The effects of the admissible corrective controls on the state variation is found by integration of variational state equations. In the impulsive-thrust case the admissible corrective controls (A9) are also impulses. Integration of Eq. (A8) using Eq. (A9) across a correction gives

$$\delta\mathbf{x}(t_k)^+ = \delta\mathbf{x}(t_k)^- + \begin{bmatrix} \omega_k\, dt_k \\ \delta\omega_k \end{bmatrix} \qquad (2)$$

where the notation

$$(t_k)^+ = \lim_{\varepsilon\to 0}(t_k + \varepsilon) \quad \text{and} \quad (t_k)^- = \lim_{\varepsilon\to 0}(t_k - \varepsilon)$$

Therefore, changing the impulse time $dt_k$ changes linearly the position variation in the direction of the nominal impulse

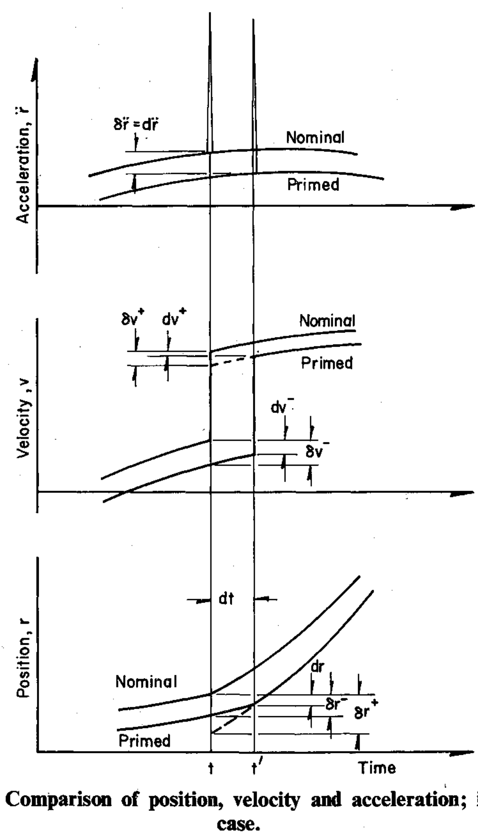


**Fig. 2 Comparison of position, velocity and acceleration; impulsive case.**

$\omega_k/|\omega_k| = \lambda_k$ and the "gain" is the magnitude of the nominal impulse $|\omega_k|$. Changing the magnitude and direction of the nominal impulse, of course, affects the corresponding velocity variation in a one-to-one manner.

In the BTA case, the result of implementing the admissible control, Eq. (A10), is more difficult to interpret because of the continuous effect of the thrust vector rotation $\delta\eta$. Integration of the variational equations across a switch time gives

$$\delta \mathbf{x}(t_k)^+ = \delta \mathbf{x}(t_k)^- + \begin{bmatrix} \mathbf{0} \\ (-1)^k a_o \lambda_k dt_k \end{bmatrix} \tag{3}$$

Changing a switch time $dt_k$ affects only the component of velocity in the direction of the nominal-thrust direction $\lambda_k$ (at time $t_k$). The "gain" is the size of the maximum thrust acceleration bound $a_0$. Across the burn, the effect of the rotation of the thrust vector, $\delta\eta$, gives

$$\delta \mathbf{x}(t_{\text{off}})^- = \delta \mathbf{x}(t_{\text{on}})^+ + a_0 \int_{t_{on}}^{t_{off}} \left\{ \Phi(t_{\text{off}}, \xi) \begin{bmatrix} \mathbf{0} \\ \mathbf{P}(\xi)\,\delta\lambda(\xi) \end{bmatrix} \right\} d\xi \tag{4}$$

where Eq. (A14) has been used to relate the optimum thrust rotation to the variational primer vector $\delta\lambda$. In general, the integral of Eq. (4) must be evaluated numerically; however, for short burn times we may approximate its value using Taylor series expansions for the time dependent variables. The transition matrix takes the form

$$\Phi(t_{\text{off}}, \xi) = \mathbf{I} - \mathbf{F}(t_{\text{off}} - \xi) + \cdots \tag{5}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{G} & \mathbf{O} \end{bmatrix}$$

To first-order in

$$\tau = t_{\text{off}} - t_{\text{on}}, \quad \text{we get}$$

$$\delta \mathbf{x}(t_{\text{off}})^- = \delta_{\mathbf{x}}(t_{\text{on}})^+ + \begin{bmatrix} \mathbf{0} \\ a_0\, \mathbf{P}_{\text{on}}\, \delta\lambda_{\text{on}}\, \tau \end{bmatrix} + \cdots \tag{6}$$

This indicates that small changes in $\delta\lambda_{\text{on}}$ affect only the velocity variation normal to the nominal-thrust direction, defined by $\mathbf{P}_{\text{on}}$, and the "gain" is the "equivalent $|\omega_k|$" $= a_0\tau$. Also to first-order we get no effect from $\delta\dot{\lambda}$.

The total change in the state variation due to the combined effort of both $dt$ and $\delta\lambda$ is for short burn times

$$\delta \mathbf{x}(t_{\text{off}})^+ = \delta \mathbf{x}(t_{\text{on}})^- + \begin{bmatrix} a_0 \lambda_{\text{on}} \tau\, dt_{\text{on}} \\ -a_0 \lambda_{\text{on}}\, dt_{\text{on}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ a_0\, \mathbf{P}_{\text{on}}\, \delta\lambda_{\text{on}} \tau \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ a_0 \lambda_{\text{off}}\, dt_{\text{off}} \end{bmatrix} + \cdots \tag{7}$$

Combining the effects of changing both switch times and rotating the thrust vector gives

$$\delta \mathbf{x}(t_{\text{off}})^+ = \delta \mathbf{x}(t_{\text{on}})^- + a_0 \tau \left\{ \lambda_{\text{on}} \begin{bmatrix} dt_{\text{on}} \\ (dt_{\text{off}} - dt_{\text{on}})/\tau \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_{\text{on}}\, \delta\lambda_{\text{on}} \end{bmatrix} \right\} \tag{8}$$

From Eq. (8) we see that $dt_{\text{on}}$ produces the only position variation to order $\tau$. This position variation is in the direction of the equivalent change in velocity $\lambda_{\text{on}}$ and the gain is the magnitude of the equivalent change in velocity $a_0\tau$. Therefore, the larger the nominal burn, the greater the effect of small changes in the switch-on time. This result parallels that of the impulsive case when the impulse time is changed. The velocity variation in Eq. (8) is seen to arise from two sources. The velocity variation in the direction of the equivalent change in velocity is linearly proportional to the normalized increase in the burn time, $(dt_{\text{off}} - dt_{\text{on}})/\tau$ and the component normal to the equivalent change in velocity results from the rotation of the thrust vector, $\mathbf{P}_{\text{on}}\,\delta\lambda_{\text{on}}$. Since $\lambda_{\text{on}}$ is a unit vector at the switch times then the gain at each guidance control is the equivalent change in velocity, $a_0\tau$.

By going to the limit as $a_0 \to \infty$ and $\tau \to 0$, we note that

$$a_0 \tau \to |\omega_k|, \quad \lambda_{\text{on}} \to \lambda_k, \quad dt_{\text{on}} \to dt_k$$
$$dt_{\text{off}} \to dt_k, \quad a_0(dt_{\text{off}} - dt_{\text{on}}) \to |\delta\omega_k| \tag{9}$$

Using these limiting values, the state variation in the bounded-thrust case Eq. (8) reduces to the state variation in the impulsive-thrust case Eq. (3).

By carrying out the terms of the contribution due to the thrust vector rotation to order of $\tau^2$, the change in state variation across a burn is

$$\delta \mathbf{x}(t_{\text{off}})^- = \delta \mathbf{x}(t_{\text{on}})^+ + a_0 \begin{bmatrix} \mathbf{0} & \mathbf{P}\tau^2/2 \\ -\mathbf{P}\tau^2/2 & \mathbf{P}\tau + \dot{\mathbf{P}}\tau^2/2 \end{bmatrix} \begin{bmatrix} -\delta\dot{\lambda}_{\text{on}} \\ \delta\lambda_{\text{on}} \end{bmatrix} \tag{10}$$

where Eq. (10) has been written in a form such that the four (3 × 3) submatrices of the transition matrix

$$\Gamma = \delta \mathbf{x}(t_{\text{off}})/\delta\psi(t_{\text{on}})$$

are shown explicitly.

The rank of the matrix $\Gamma$ equals the number of independent controllable states available by changing the thrust vector direction and rate of change of direction. From its approximate form given by Eq. (10) the rank of $\Gamma$ is only 2 to order $\tau$ and at most 5 to order $\tau^2$. Numerically, no case has yet been found where $\Gamma$ had maximal rank (rank = 6). It is understandable that it does not have maximal rank in that if it did this would imply that any final state could be attained by simple rotation of the thrust vector without need for more propellant (longer thrust time).

The optimality condition that between switch points

$$\psi^T \delta \mathbf{x} = \text{const} \tag{11}$$

requires that those variations in the direction of the adjoint vector not be controlled by variations in the thrust vector direction. These variations are controlled by variations in the switch times because $\psi^T \delta \mathbf{x}$ is not constant across switch points. Specifically, using Eq. (3) we get

$$\psi^T \delta \mathbf{x} = (-1)^k a_0\, dt_k \qquad \text{(across switch points)} \tag{12}$$

By considering the combined effect of both the switch-on and the switch-off points we get for each burn

$$\psi^T \delta \mathbf{x} = a_0(dt_{\text{off}} - dt_{\text{on}}) \qquad \text{(across a burn)} \tag{13}$$

In the limiting behavior of Eq. (9), the BTA case given by Eq. (13) reduces to the jump in $\psi^T \delta \mathbf{x}$ across impulses given by

$$\psi^T \delta \mathbf{x} = \lambda_k^T\, \delta\omega_k \qquad \text{(across an impulse)} \tag{14}$$

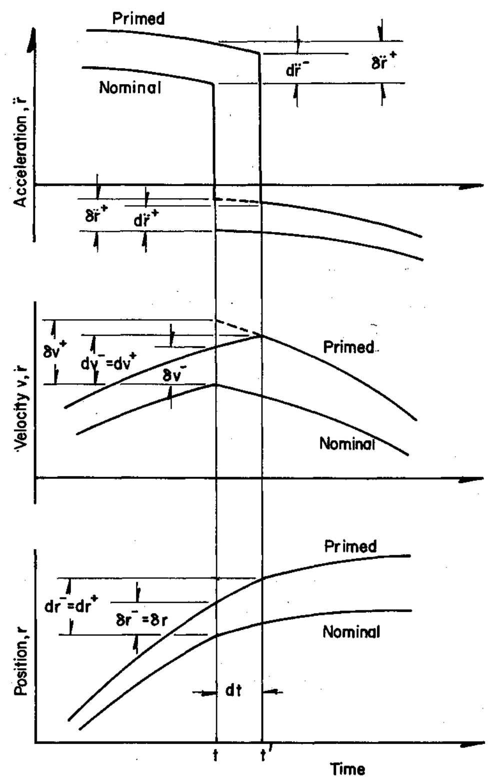


**Fig. 3 Contemporaneous and skewed variations, bounded-thrust case.**

**Table 1 Boundary conditions for test case F-2 two-dimensional Earth-Mars rendezvous trajectory**

Initial adjoint

| $-\dot{\lambda}(1)$ | $-\dot{\lambda}(2)$ | $-\dot{\lambda}(3)$ | $\lambda(1)$ | $\lambda(2)$ | $\lambda(3)$ |
|---|---|---|---|---|---|
| −0.78714 | −0.62846 | 0.0 | −0.96294 | −0.07253 | 0.0 |

Coordinates

| Time (days) | r(1) | r(2) | r(3) | v(1) | v(2) | v(3) |
|---|---|---|---|---|---|---|
| Departure | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| t = 29.88 | 0.83874 | 0.49198 | 0.0 | −0.59538 | 0.86964 | 0.0 |
| t = 50.24 | 0.57268 | 0.75791 | 0.0 | −0.93014 | 0.63520 | 0.0 |
| t = 99.99 | −0.36562 | 0.93202 | 0.0 | −1.08237 | −0.24061 | 0.0 |
| Arrival | −1.17700 | −0.96760 | 0.0 | 0.51448 | −0.62580 | 0.0 |

Nominal thrust on-off program

| Time (days) | Switch | Radius (A.U.) | Travel angle (deg) |
|---|---|---|---|
| 0 (departure) | on | 1.0 | 0.0 |
| 22.09 | off | 0.983 | 22.15 |
| 37.45 | on | 0.963 | 38.72 |
| 58.37 | off | 0.944 | 61.49 |
| $T = 62.82$ days: perihelion; $R = 0.943$ | | | |
| 215.97 | on | 1.457 | 197.69 |
| 258.87 | off | 1.524 | 219.43 |

**Independent Degrees of Freedom**

In the impulsive-thrust case the number of independent guidance controls available at each impulse is obviously equal to four: $dt_k$ and the three independent components of $\delta\omega_k$. In the BTA case the independent controls are not so readily identified. Since $\delta\eta(t)$ is a continuous function of time at first there seems to be an infinite number of degrees of freedom. However, optimality requires that $\delta\eta^*(t) = \mathbf{P}\delta\lambda$ and since $\delta\lambda(t)$ is described by a second-order differential equation, the maximum number of independent controls is reduced to six,

$$\delta\psi_0 = \begin{bmatrix} -\delta\dot{\lambda}_0 \\ \delta\lambda_0 \end{bmatrix}$$

plus the two switch times. Because of Eq. (11) the number is further reduced by one.

By considering the effectiveness of the guidance corrections given by Eq. (10) we see that for short duration burns the number of independent guidance controls are reduced to four: the two components of $\delta\lambda$ normal to $\lambda$ and the two changes in the switch times. This assumes that $\delta\lambda$ and $\delta\dot{\lambda}$ are of the same order. Even for longer burns, the correction capability in the three components of velocity and the one component of position in the direction of $\lambda$ are of order $a_0\,\tau\,\delta\lambda$ and $a_0\,\tau\,dt$, respectively, while the other components of the position variation are of the lower order $a_0\,\tau^2\,\delta\lambda + \mathcal{O}(\tau^3\,\delta\dot{\lambda})$.

Basically, the result is that the most useful corrections in the BTA case parallel those of the impulsive-thrust case. The predominant effect of thrusting is to impart a change on the spacecrafts velocity vector. Optimal coasting is used to convert the change in velocity into a change in position. In the guidance law, the changing of the thrust parameters, direction, $\delta\eta$, and burn duration $dt_{\text{off}} - dt_{\text{on}}$, are most effective in correcting the instantaneous velocity variation. The changing of the location of the beginning of the burn by $dt_{\text{on}}$ changes the coasting time and, therefore, the position.

## One-Burn Trajectory

In order to develop a feedback guidance law for all portions of a trajectory we need to consider the case of one-burn trajectories. Since all multiburn trajectories reduce to one-burn trajectories after some time, guidance along such trajectories is of primary importance.

In the impulsive-thrust case the optimal guidance law discussed in the previous sections is no longer applicable. The guidance law is uncontrollable because the admissible guidance controls have an insufficient number of degrees of freedom. If the impulse is an interior impulse, only the position variation normal to $\lambda_k$ and the velocity variation at $t_k$ can be changed using the admissible guidance controls. Any position variation normal to the primer vector at the nominal impulse time can be nulled only by adding small guidance impulses.[11] For terminal impulses, all three components of the projected final-time position variation must be nulled by using additional impulses. Now that additional impulses must be included in the allowable control set the optimal guidance problem is separated into two problems. The first is to find the number, timing, magnitude and direction of the additional impulse $\delta v_i(t_i)\; i = 1,\cdots,m$ such that only the projected final position variation (for the terminal-impulse case) is nulled while minimizing

$$\sum_{i=1}^{m} |\delta\mathbf{v}_i|$$

i.e., find:

$$\delta\mathbf{v}_i, \quad t_i \quad \text{and} \quad m$$

that minimizes:

$$\sum_{i=1}^{m} |\delta\mathbf{v}_i|$$

subject to

$$= \mathbf{F}\,\delta\mathbf{x} + \mathbf{D}\,\delta\mathbf{v}_i\,\delta(t - t_i)$$

with $\delta\mathbf{x}_0$ given and/or (15)

$$\delta\mathbf{r}(t_f) = 0, \quad \delta\mathbf{v}(t_f) \text{ free} \quad \text{(terminal impulse)}$$

$$\mathbf{P}\delta\mathbf{r}(t_k) = 0; \quad \delta\mathbf{v}(t_k), \lambda^T\,\delta\mathbf{r}(t_k) \text{ free} \quad \text{(interior impulse)}$$

These additional impulses will optimally null the state errors that are not controllable by the augmentation of the final impulse. The second problem is simply to choose the allowable augmentations of the nominal impulse to satisfy $\delta\mathbf{v}(t_f) = 0$ (plus $\lambda^T\,\delta\mathbf{r}(t_k) = 0$ for an interior impulse). In most practical applications, the last coast-impulse arc is short (less than 180°). For such cases problem Eq. (15) will probably call for the addition of one additional guidance impulse as soon as possible. In this case the problem of determining the additional guidance impulse reduces to Battin's[4] fixed-time-of-arrival guidance for cases where the nominal impulse is a terminal impulse and to the variable-time-of-arrival guidance problem for cases where the nominal impulse is an interior impulse.

Table 2 Data for the curves presented in Fig. 4

| Components of primer vector | Nominal | 4.1 | 4.2 | 4.3 |
|---|---|---|---|---|
| | | Case Number | | |
| | 0 | Nonzero final state variation: $\delta v_f(2) = 10^{-4}$ | $\delta v_f(2) = 10^{-6}$ | $\delta v_f(3) = 10^{-4}$ (out of plane) |
| $dt_0$ | 0 | 0 | 0 | 0 |
| $dt_1$ | 0 | $-0.270 \times 10^{-4}$ | $0.766 \times 10^{-5}$ | $6.9 \times 10^{-6}$ |
| $\lambda_1(t_0)$ | −0.53792 | −0.79977 | −0.11976 | −0.56868 |
| $\lambda_2(t_0)$ | −0.03741 | 0.02269 | −0.14907 | −0.02192 |
| $\lambda_3(t_0)$ | 0.0 | 0.0 | 0.0 | −0.00707 |
| $\lambda_1(t_1)$ | 0.79631 | 0.79608 | 0.79646 | 0.79630 |
| $\lambda_2(t_1)$ | −0.07773 | −0.07997 | −0.07610 | −0.07784 |
| $\lambda_3(t_1)$ | 0.0 | 0.0 | 0.0 | −0.00123 |

In the bounded-thrust case the coast-burn (or coast-burn-coast) trajectory is also uncontrollable without the extension of the admissible control set to include the addition of short duration burns. This is in direct parallel to the one-impulse case. For long duration burns, it is not clear qualitatively which components of the first-order final state variation are uncontrollable; however, they can be determined quantitatively for each particular case. As in the impulsive case, the additional burns should be chosen to minimize the propellant required to null-out the inaccessible final state variations. The remaining accessible final state variations are to be nulled by the augmentation of the nominal burn.

If the nominal burn is of short duration then, from Eq. (7) small rotations of the thrust vector and small changes in the switch times can be used to null-out the final-time velocity vector variation and the component of the position vector in the direction of $\lambda(t_f)$. The remaining components of position must be nulled by the addition of at most two short-duration burns. For coast-thrust trajectories having short travel arcs, the best place to add a burn will most likely be at the initial time. Using the short-burn-time assumption the direction and duration of this burn is given by

$$a_0 \lambda \, dt = \Phi_{12}(t_0, t_f)^{-1} \mathbf{P}_f(\Phi_{11}(t_0, t_f) \delta \mathbf{r}_0 + \Phi_{12}(t_0, t_f) \delta \mathbf{v}_0) \quad (16)$$

After implementing Eq. (16), Eq. (10) can be solved for the augmentation of the nominal burn that will null-out the remaining components of the projected final-time state variations. This is the finite thrust analogue to Battin's fixed-time-of-arrival guidance.

## Numerical Results

In Fig. 4 and 5 data are presented from a numerical study of optimal trajectories in the neighborhood of the final coast-burn portion of a typical Earth-Mars rendezvous. The data for the nominal trajectory is presented in Table 1 and the heliocentric map for the mission is presented in Fig. 1. The guidance analysis chooses the 100th day into the flight as the "initial time." Figure 4 considers initial state perturbations of the type that if they were left uncorrected, they would result in final-time velocity variations as indicated. The resulting optimal trajectories from the perturbed initial conditions are coast-burn trajectories for the cases presented. The guidance corrections that drive the spacecraft from the perturbed initial state $\delta \mathbf{x}_o$ to the nominal final state $\delta \mathbf{x}_f = 0$ are accomplished by the augmentation of the nominal burn. For case 4.3, where only the projected final-time, out-of-plane velocity is perturbed, note that the change in thrust-on time given in Table 2 is much smaller than for a equal sized perturbation in the in-plane velocity component, case 4.1. Also note the disagreement at $t_o$ and close agreement at $t_1$ in the various values of the components of the primer vector $\lambda$. Additional numerical results have indicated that for smaller sized perturbations the neighboring optimal trajectory is a coast-burn trajectory. This is illustrated by comparing cases 4.1 and 4.2. However, for larger variations the neighboring optimal trajectory becomes a two-burn trajectory. The size of the perturbation relative to the tolerance in the final state conditions seems to be the important factor. The switch function time curve for case 4.1 indicates that for a slightly larger perturbation an initial burn may appear. This was found to be the case for a $\delta \mathbf{x}_f(1) = 10^{-3}$. The tolerance on the final state was $10^{-8}$ for all cases studied.

For cases where final-time position variations are nonzero, two-burn neighboring trajectories appear for smaller sized perturbations. Three cases are presented in Fig. 5. The effect of the added burn is to null-out the final-time position variation. Cases 5.1 and 5.2 provide a comparison of asymmetrical perturbations. Note the resulting asymmetrical thrust direction at $t_o$ given in Table 3 and the asymmetrical change in the nominal thrust-on time. Case 5.3 considers an out-of-plane position variation.

## Conclusion

The purpose of this paper has been to link together the minimum propellant impulsive and bounded-thrust guidance

Table 3 Data for the curves presented in Fig. 5

| Components of primer vector | Nominal | 5.1 | 5.2 | 5.3 |
|---|---|---|---|---|
| | | Case Number | | |
| | 0 | Final state variation: $\delta r_f(1) = 10^{-4}$ | $\delta r_f(1) = -10^{-4}$ | $\delta r_f(3) = 10^{-3}$ (out of plane) |
| $dt_0$ | 0 | $1.333 \times 10^{-4}$ | $1.365 \times 10^{-4}$ | $2.2 \times 10^{-5}$ |
| $dt_1$ | 0 | $5.307 \times 10^{-4}$ | $-5.195 \times 10^{-4}$ | $-3.109 \times 10^{-5}$ |
| $\lambda_1(t_0)$ | −0.53792 | 0.71407 | −0.79973 | −0.77725 |
| $\lambda_2(t_0)$ | −0.03741 | −0.36109 | 0.02715 | 0.02626 |
| $\lambda_3(t_0)$ | 0.0 | 0.0 | 0.0 | 0.18802 |
| $\lambda_1(t_1)$ | 0.79631 | 0.79694 | 0.79612 | 0.79495 |
| $\lambda_2(t_1)$ | −0.07773 | −0.07098 | −0.07958 | −0.07864 |
| $\lambda_3(t_1)$ | 0.0 | 0.0 | 0.0 | 0.04481 |

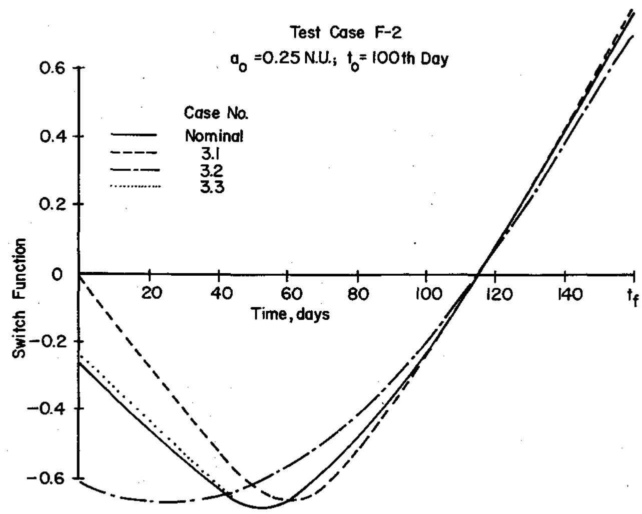


Fig. 4 **Switch function vs time for optimal trajectories having slightly different initial state than the coast-burn nominal trajectory; final-time velocity variations only; final state tolerance = $10^{-8}$.**

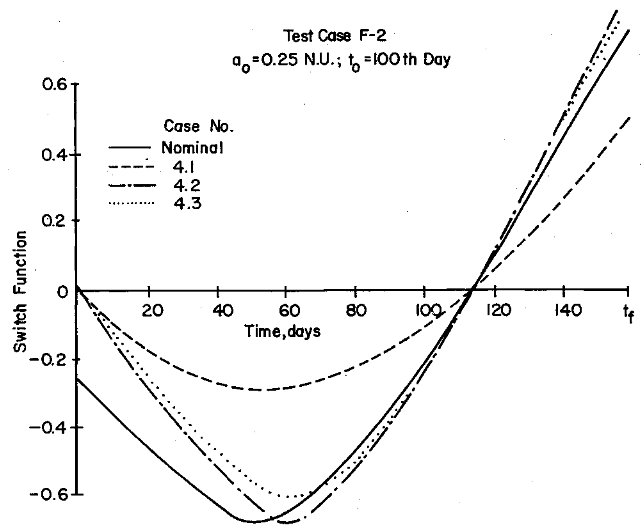


Fig. 5 **Switch function vs time for optimal trajectories having slightly different initial state than the coast-burn nominal trajectory; final-time position variations only; final state tolerance = $10^{-8}$.**

problems. Using this link it is possible to gain some qualitative knowledge about the bounded-thrust case in terms of the more easily understood impulsive-thrust case. This becomes especially useful when considering the guidance problem in the neighborhood of coast-burn trajectories. In such cases, quantitative solutions to the optimal guidance law in terms of a neighboring extremal formulation are found to be uncontrollable. A method of getting around this difficulty, at least for short duration burns, is to reformulate the bounded-thrust guidance problem in terms of the one-impulse guidance solution. Numerical investigations of the bounded-thrust case indicate that good results can be expected from such a reformulation.

## Appendix A: A Unified Development of Minimum-Propellant Guidance Laws

The equations of spacecraft motion in state space notation are

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{D}\,\mathbf{u} \tag{A1}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{r} \end{bmatrix} ; \quad \mathbf{f} = \begin{bmatrix} \mathbf{v} \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 0 \\ \mathbf{I}_3 \end{bmatrix}$$

Physically $\mathbf{u}$ is the acceleration due to thrust

$$\mathbf{u} = (a/m)\hat{\eta} \tag{A2}$$

where $\hat{\eta}$ is the unit thrust vector direction, $a$ is the thrust of the rocket which is assumed to be bounded

$$0 \leq a \leq a_o \tag{A3}$$

and $m$ is the mass of the rocket. For simplicity, this development is restricted to the case where the mass flow rate

$$m = -a/c \tag{A4}$$

is sufficiently large to warrant the assumption that the mass of the rocket remains constant. Equation (A3) then represents a bounded thrust acceleration (BTA) constraint.

The adjoint system correspondent to Eq. (A1) is

$$\dot{\psi} = -\mathbf{F}^T\psi \tag{A5}$$

where $\mathbf{F} = \partial\mathbf{f}/\partial\mathbf{x}$ is a $(6 \times 6)$ matrix and

$$\psi = \begin{bmatrix} -\dot{\lambda} \\ \lambda \end{bmatrix}$$

is a 6-vector.

For simplicity, we consider only fixed-time rendezvous trajectories, that is $\mathbf{x}(0)$ and $\mathbf{x}(t_f)$ given. Extension to other types of boundary conditions is straightforward.

For the case where $a_o$ of Eq. (A3) becomes unbounded the optimal control $\mathbf{u}^*$ that maximizes the final spacecraft mass and satisfies the boundary condition takes the form of a sequence of impulses, i.e.,

$$\mathbf{u}^*(t) = \sum_k |\omega_k| \lambda(t_k)\,\delta(t - t_k) \quad k = 1,\ldots,n \tag{A6}$$

where $|\omega_k|$ is the magnitude of the $k$th of $n$ impulses. The impulse times $t_k$ of the Dirac delta function $\delta(t - t_k)$ are at the times where the magnitude of the primer vector,

$$p_k^2 = \lambda^T{}_k \lambda_k = 1.$$

The subscript $k$ denotes evaluation at time $t_k$. Additional necessary conditions satisfied by the primer magnitude are $p(t)$ continuous and $p(t) \leqq 1$ for all $0 \leqq t \leqq t_f$.

For the cases when the thrust-acceleration bound $a_o$ is finite the optimal control $\mathbf{u}^*$ that minimizes $\int a dt$ (equivalent to maximizing final mass is)

$$\mathbf{u}^*(t) = \begin{cases} 0 & \text{if} \quad s = p - 1 < 0 \\ a_o \dfrac{\lambda}{p} & \quad\;\; s = p - 1 > 0 \end{cases} \tag{A7}$$

The optimal controls $\mathbf{u}^*(t)$ given in Eq. (A6) and (A7) define the nominal trajectory. The optimal guidance problem is to determine the corrections to the nominal thrust program,

$$\delta\mathbf{u}^*(t) \equiv \mathbf{u}_{\text{perturbed}}{}^*(t) - \mathbf{u}_{\text{nominal}}{}^*(t),$$

that drive the spacecraft from a perturbed initial state $\delta\mathbf{x}(t_o)$ to the nominal final state $\delta\mathbf{x}(t_f) = 0$ and maximizes the final spacecraft mass. Using a neighboring extremal point of view, the effect of various guidance stratigies can be evaluated and in the multiburn case, the formulation can be placed in the form of an accessory minimum problem: minimize a quadratic form subject to linear constraints.

The first-order variational equation

$$\delta\dot{\mathbf{x}} = \mathbf{F}\,\delta\mathbf{x} + \mathbf{D}\,\delta\mathbf{u} \tag{A8}$$

measures the "effect" of the corrective control $\delta\mathbf{u}$. In the impulsive-thrust case, the possible changes in the nominal-thrust program are: 1) the addition of small impulses at any time during the flight; and 2) small changes in the timing of the nominal impulses (large changes in the above corrections would produce, through Eq. (A8), much larger than desired corrections). In the bounded-thrust acceleration (BTA) case the possible changes in the nominal thrust program are: 1) the addition of "short" corrective thrust/coast periods during nominal coast/thrust periods, 2) "small" reductions in the thrust magnitude during the nominal thrusts (increasing the thrust magnitude is disallowed because of the thrust bound); and 3) changes in the direction of the thrust vector.

By implementing the various corrective controls and comparing the cost (final mass) of the perturbed trajectory to that of the nominal trajectory it can be shown[1-3] that some corrections increase the cost to first order, i.e., the cost increases linearly with the magnitude of the corrective control. The other corrections provide a quadratic (second order) increase in cost. Therefore, to minimize the cost the controls that increase the cost to first order are disallowed and the others are chosen to minimize the second-order increase in cost while satisfying the boundary conditions. Specifically, in the impulsive-thrust case the addition of impulses at all times except the nominal impulse time increases the first-order cost. The addition of small impulses at the nominal impulse times and small changes in the impulse times for interior (not at a fixed time boundary) impulses contribute to the cost only to second order. The addition of small impulses at nominal impulse times can better be interpreted as small changes in the magnitude and direction of the nominal impulses. Thus, in the impulsive-thrust case truly new impulses are disallowed and

$$\mathbf{D}\,\delta\mathbf{u}(t) = \sum_k \begin{bmatrix}\omega_k\,dt_k\\ \delta\omega_k\end{bmatrix}\delta(t-t_k) \tag{A9}$$

where $\delta\omega_k$ $(3\times 1)$ represents the change in magnitude and direction of the nominal impulse and $dt_k$ (scalar) is the change in the timing of the nominal impulse.

For the BTA case, only the addition of short burn/coasts in the neighborhood of the nominal switch times (small changes in the switch times) and small rotations of the nominal thrust direction do not increase the first-order cost. These admissible corrective controls are written

$$\delta\mathbf{u}(t) = a_o\,\mathbf{P}\,\delta\eta + \sum_k(-1)^k a_o\,\eta_k\,dt_k\,\delta(t-t_k) \tag{A10}$$

where the first term is the contribution of small changes in the thrust direction and the second term is the change in the switch time.

The optimal guidance law $\delta\mathbf{u}^*(t)$ is found by minimizing: the second variation in cost $\delta^2 J$; subject to: 1) linear variational state equations, Eq. (A8), and 2) boundary conditions, $\delta\mathbf{u}_o$ given, $\delta\mathbf{x}_f = 0$.

The form of $\delta^2 J$ that minimizes the total propellant requirements is for impulsive thrust

$$\delta^2 J = \left[\frac{1}{2}\sum_k \delta\mathbf{u}^T{}_k\,\mathbf{R}_k\,\delta\mathbf{u}_k - \int_{t_o}^{t_f}\{\delta\mathbf{x}^T\mathbf{H}_{xx}\,\delta\mathbf{x}\}\,dt\right] \tag{A11}$$

where

$$\mathbf{R}_k = \begin{bmatrix}|\omega_k|\,\lambda^T{}_k\,\mathbf{G}_k\,\lambda_k & \dot{\lambda}_k{}^T\\ \dot{\lambda}_k & \mathbf{P}_k\end{bmatrix} \qquad \mathbf{G}_k = \left.\frac{\partial\mathbf{q}}{\partial r}\right|_{t_k} \quad (3\times 3)$$

$$\mathbf{P}_k = [\mathbf{I}_3 - \lambda_k\lambda_k{}^T]/|\omega_k| \qquad (3\times 3),\ \text{rank} = 2$$

$$\delta\mathbf{x}^T\mathbf{H}_{xx}\,\delta\mathbf{x} = \delta\mathbf{r}^T\frac{\partial^2(\lambda^T\mathbf{q})}{\partial\mathbf{r}^2}\delta\mathbf{r}$$

and for BTA

$$\delta^2 J = -\tfrac{1}{2}\sum_k(-1)^k a_o\,\dot{p}_k\,dt_k{}^2 - \tfrac{1}{2}\int_{t_o}^{t_f}\delta^2\mathbf{H}\,dt \tag{A12}$$

where

$$\delta^2\mathbf{H} = [\delta\mathbf{x}^T\ \delta\psi^T]\begin{bmatrix}\mathbf{H}_{xx} & \mathbf{H}_{x\psi}\\ \mathbf{H}_{\psi x} & \mathbf{H}_{\psi\psi}\end{bmatrix}\begin{bmatrix}\delta\mathbf{x}\\ \delta\psi\end{bmatrix}$$

$\mathbf{H}$ is the Hamiltonian and $\delta\psi$ is the adjoint variable associated with Eq. (A8). For both the impulsive and BTA cases $\delta\psi$ is described by

$$\delta\dot{\psi} = - \quad {}^T\delta\psi - \delta\mathbf{x}^T\mathbf{H}_{xx}\,\delta\mathbf{x} \tag{A13}$$

$\delta^2 J$ is minimized with respect to $\delta\eta$ by choosing

$$\delta\eta^*(t) = \mathbf{H}_{\eta\eta}{}^{-1}\,\mathbf{H}_{\eta\psi}\,\delta\psi = \mathbf{P}\,\delta\lambda(t) \tag{A14}$$

Equation (A14) in conjunction with Eq. (A13) reduces the problem of finding the vector function $\delta\eta^*(t)$ to the problem of finding the vector $\delta\psi(t_o)$.

Using Eq. (A14) and the transition matrix, $\Lambda(t,\tau)$, solution to Eqs. (A8) and (A13)

$$\begin{bmatrix}\delta\mathbf{x}(t)\\ \delta\psi(t)\end{bmatrix} = \Lambda(t,\tau)\begin{bmatrix}\delta\mathbf{x}(\tau)\\ \delta\psi(\tau)\end{bmatrix} + \begin{bmatrix}\mathbf{B}_k\\ \mathbf{0}\end{bmatrix}dt_k \tag{A15}$$

the integrals of Eqs. (A11) and (A12) may be evaluated analytically. Thus, in both cases it is possible to write

$$\delta^2 J = [\delta\mathbf{x}_o^T\ \delta\mathbf{u}^T]\begin{bmatrix}\mathbf{Q}_{11} & \mathbf{Q}_{12}\\ \mathbf{Q}_{21} & \mathbf{Q}_{22}\end{bmatrix}\begin{bmatrix}\delta\mathbf{x}_o\\ \delta\mathbf{u}\end{bmatrix}$$

Equation (A15) is also used to write $\delta\mathbf{x}_f$ in terms of $\delta\mathbf{x}_o$ and $\delta\mathbf{u}$. Requiring that $\delta\mathbf{x}_f = 0$ specifies 6 of the controls $\delta\mathbf{u}'$

$$\delta\mathbf{u}' = \mathbf{C}^{-1}[\Phi(t_f,t_o)\,\delta\mathbf{x}_o + \textstyle\sum\Phi(t_f,t_k)\,\mathbf{B}_k\,\delta\mathbf{u}_k{}''] \tag{A17}$$

where

$$\delta\mathbf{u} = \begin{bmatrix}\delta\mathbf{u}'\\ \delta\mathbf{u}''\end{bmatrix}$$

$$\Phi(t_f,t_o) = \partial\mathbf{x}(t_f)/\partial\mathbf{x}(t_o)$$

$$\mathbf{B}_k = \begin{bmatrix}-\lambda_k & |\omega_k| & \mathbf{0}\\ \mathbf{0} & & \mathbf{I}_3\end{bmatrix} \quad \text{impulsive case}$$

$$= (-1)^{k+1}\,\mathbf{D}\,a_o\,\lambda_k/p_k \quad \text{BTA case}$$

and $\mathbf{C}$ is the controllability matrix. For impulsive thrust $\mathbf{C}^{-1}$ is Battins[4] fixed-time-of-arrival matrix.

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Using Eq. (A17) reduces Eq. (A16) to an unconstrained quadratic form in the variables $\delta\mathbf{x}_o$ and $\delta\mathbf{x}''$

$$J = [\delta\mathbf{x}_o^T\ \delta\mathbf{u}''^T]\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\delta\mathbf{x}_o\\ \delta\mathbf{u}''\end{bmatrix} \tag{A18}$$

Optimal values of $\delta\mathbf{u}''^*$ result from simply

$$\partial\,\delta^2 J/\partial\,\delta\mathbf{u}'' = 0$$

$$\delta\mathbf{u}''^* = -\mathbf{A}_{22}^{-1}\,\mathbf{A}_{21}\,\delta\mathbf{x}_o \tag{A19}$$

The remaining optimal values of $\delta\mathbf{u}^*$ result from introducing Eq. (A19) into Eq. (A17).

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